Volterra $\overline{GZ}$ approach - A new method to accurately calculate the nonlinear and time varying roll restoring arm of ships in irregular longitudinal seas

Abhilash Somayajula [a] and Jeffrey Falzarano[a]

[a] Department of Ocean Engineering, Texas A&M University, College Station, TX - 77843, United States

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ABSTRACT
With the several incidents of large amplitude roll motion of ships observed at sea over the past two decades, dynamic stability of modern ship types has assumed significant importance. The move by the International Maritime Organization to develop a second generation level-2 criteria has resulted in a considerable attention to the development of simplified analytical models for roll motion. Especially for the problem of parametric roll, there is still no analytical model which accurately captures the nonlinear roll restoring stiffness taking into account the combined effect of incident waves and dynamic heave and pitch motions of the vessel. This paper extends our previous work (Somayajula and Falzarano 2017) to accurately capture both the linear and higher order time varying roll stiffness coefficients through an analytical model while taking into account incident waves and dynamic motion of the vessel. This approach called the Volterra $\overline{GZ}$ method is compared against the existing simplified models and a nonlinear time domain simulation program to demonstrate better agreement of the proposed approach.

KEYWORDS
Volterra GZ; Parametric roll; Analytical roll model; Nonlinear roll restoring arm; Single degree of freedom model; Grim effective wave; SIMDYN; Nonlinear time domain simulation

1. Introduction

Since the accident of APL China in 1998, accurate modeling of roll restoring moment or roll restoring arm of ships in head seas has received wide attention from several research groups across the world. Over the past two decades, several models have been proposed to simplify the problem for further analytical study. However, the accurate modeling of parametric roll continues to be an active area of research and the academic community has not yet agreed upon a single accepted approach to calculate the roll restoring arm in longitudinal seas.

Paulling (1961) was one of the earliest researchers to extensively study this phenomenon. Paulling (1961) computed righting arm curves in longitudinal regular waves and validated them against experiments. Although he was able to derive an analytical

A. Somayajula. Email: s.abhilash89@gmail.com Website: https://sites.google.com/site/sabhilash89
J. Falzarano. Email: jfalzarano@civil.tamu.edu Website: http://people.tamu.edu/~jfalzarano

method for calculation of metacentric height $\overline{GM}(t)$, a similar analytical model could not be obtained for the righting arm $\overline{GZ}(t)$. Neves and Rodriguez (2005) (also see Neves and Rodríguez 2007) derived a more rigorous third order analytical model for roll restoring arm based on an approach similar to Paulling. However, the downside of this model is its significant complexity and the computation of several geometry based coefficients (Bulian 2006). A simplified model and a panel method code were investigated by Spyrou et al. (2008). It was concluded that the analytical formulae developed for a Matheiu-type system did not quite capture the reality of non-harmonic variation of $\overline{GZ}$ in waves. A Mathieu-type model and a Hill's equation based model for parametric roll were compared by Moideen et al. (2012, 2013). It was found that a Hill's equation model was more accurate as it accounted for the non-harmonic variation of metacentric height even when the ship is subjected to regular waves. Including the linearized quadratic roll damping Moideen et al. (2012) extended the traditional Ince-Strutt diagram into a 3-D plot providing the bounded roll motion amplitude. However, all of the studies mentioned above are limited to regular incident waves and cannot be easily extended to irregular waves.

A popular approach to extend regular wave analysis to irregular waves using the concept of Grim effective wave (originally studied by Grim (1961)) has been investigated by many researchers including Bulian (2008); Umeda et al. (2004); Hashimoto et al. (2006) and Dostal et al. (2012). The Grim effective wave approximates the spatial wave elevation at each time by a least square fit regular wave whose wavelength is equal to the ship's length. This approach has received wide popularity due to its ease in reducing the problem in irregular seas to that in regular waves. However, a comparative study of various simplified models for parametric roll undertaken by Somayajula and Falzarano (2017) demonstrated that the Grim effective wave models do not agree very well with the nonlinear time domain simulation program which has been validated against experiments. Bulian et al. (2008) studied the Grim effective wave models extensively and concluded that the match with experiments was only somewhat reasonable.

A different alternative to the Grim's effective wave approach to calculate the roll restoring stiffness in irregular waves was investigated originally by Palmquist (1994) and later formalized by Hua et al. (1999). This approach modeled the metacentric height $\overline{GM}$ in waves using a series of Volterra transfer functions. Although this approach does not capture of variation of nonlinear roll restoring arm $\overline{GZ}$ in irregular waves, it has still received significant interest. Moideen et al. (2014) and Somayajula et al. (2014) applied the Volterra $\overline{GM}$ approach to simulate parametric roll of a C11 container ship in head seas. This approach has been further extended by Somayajula and Falzarano (2014, 2015b) to include a time invariant cubic restoring arm. Hua's model (Hua et al. 1999) includes a time varying $\overline{KG}$ contribution to the time variation of $\overline{GM}$. However, it is quite well known that $\overline{KG}$ of a ship (distance between the keel and the center of gravity) is time invariant. Somayajula and Falzarano (2017) introduced this correction and compared this model against both the Grim effective wave model and a nonlinear time domain simulation program described in detail in Somayajula and Falzarano (2015a). This study demonstrated that both the Volterra $\overline{GM}$ and Grim effective wave approaches over predict the roll response when compared against a time domain simulation program which was validated with experiments.

Simple and yet accurate modeling of parametric roll of ships in irregular seas continues to be challenge and is still an active area of research. In this paper a new modeling approach - the Volterra $\overline{GZ}$ method - is proposed to overcome some of the limitations of the existing approaches. This method extends the Volterra $\overline{GM}$ approach to analytically model the time varying restoring arm $\overline{GZ}$ in irregular waves without imposing a harmonic fit approximation like Grim effective wave while accounting for higher order time varying restoring stiffness terms.

2. Single Degree of Freedom Roll Equation of Motion

For a ship subjected to head on waves, the roll mode of motion may be assumed to be approximately decoupled from the other modes (Webster 1989). The 1-DOF roll equation of motion can then be expressed as shown in (1). Both the linear and quadratic viscous roll damping coefficients are obtained using the standard Himeno (1981) approach and the details are described in Falzarano et al. (2015).

$$[I_{44} + A_{44}(\infty)]\ddot{\phi} + \int_{-\infty}^{t} K_{44}(t-\tau)\dot{\phi}(\tau)d\tau + B_1\dot{\phi} + B_2\dot{\phi}|\dot{\phi}| + C_{44}(t,\phi) = 0 \quad (1)$$

where

I_{44} is the roll mass moment of inertia
$A_{44}(\infty)$ is the infinite frequency roll added moment of inertia
$K_{44}(\tau)$ is the roll impulse response function (IRF) / roll retardation function
B_1 is the linear roll damping coefficient due to viscous effects
B_2 is the quadratic roll damping coefficient due to viscous effects
$C_{44}(t,\phi)$ is the nonlinear instantaneous roll restoring moment including the dynamic effects due to instantaneous wave elevation and the heave and pitch motions

The roll impulse response function $K_{44}(\tau)$ is computed from the encounter frequency dependent roll radiation damping $B_{44}(\omega_e)$ as shown in (2). The added mass and radiation damping at various frequencies including the infinite frequency are obtained using an in-house frequency domain panel method code (Guha and Falzarano 2016).

$$K_{44}(\tau) = \frac{2}{\pi}\int_{0}^{\infty} [B_{44}(\omega_e) - B_{44}(\infty)]\cos(\omega_e\tau)d\omega_e \quad (2)$$

In a previous investigation into the phenomenon of parametric roll using nonlinear time domain simulation (Somayajula and Falzarano 2015a) it was observed that even after the onset of unstable roll motion, the heave and pitch motions of the vessel closely follow the linear theory. However, the heave and pitch motions of the vessel lead to a significant time varying change of the instantaneous underwater displacement which must be captured accurately to model the parametric excitation responsible for large amplitude unstable roll motion. It is noted that the time variation of the stiffness term $C_{44}(t,\phi)$ also implicitly includes the effects of instantaneous heave $\xi_3(t)$ and pitch $\xi_5(t)$ motions of the vessel.

3. Roll Restoring Moment

The roll restoring moment of a statically balanced ship can be expressed as the product of the weight $W = \rho g \nabla_0$ and the restoring arm $\overline{GZ}$ where ∇_0 is the calm water displacement of the vessel. In case of a static equilibrium, the moment due to the

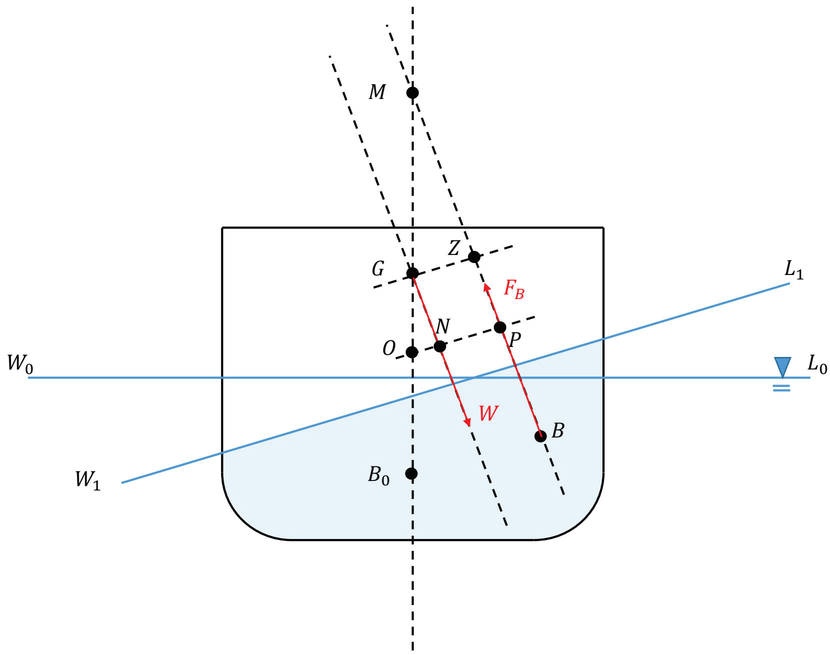


Figure 1.: Roll restoring moment

weight and the restoring moment are equal and opposite. However, in a dynamic equilibrium as described by (1), there are other forces such as inertial, damping and external forces acting on the system. This implies that the buoyancy force is no longer equal in magnitude to the weight.

Consider a transverse section of a ship at a distance x from the origin as shown in Figure 1. Let $\overline{W_0L_0}$ be the calm water line and $\overline{W_1L_1}$ be the relative waterline (including the effects of the incident wave, heave and pitch motions) for an instantaneous roll angle ϕ. It is assumed that the rotation is considered about the longitudinal axis of the body fixed frame with its origin at O. Due to the change in underwater hull form, the center of buoyancy shifts from B_0 to B. The buoyancy force $F_B = \rho g \nabla(t, \phi)$ now acts along the line $\overline{BM}$. From Figure 1 the instantaneous moment about O is given by (3).

$$\begin{aligned} C_{44}(t,\phi) &= F_B \times \overline{OP} - W \times \overline{ON} \\ &= \rho g \nabla(t,\phi)\overline{OP} - \rho g \nabla_0 \overline{ON} \\ &= \rho g \nabla_0 \left[\overline{OP} \frac{\nabla(t,\phi)}{\nabla_0} - \overline{ON} \right] \\ &= \rho g \nabla_0 \overline{GZ}_{\text{eff}}(t,\phi) \end{aligned} \tag{3}$$

$$\overline{GZ}_{\text{eff}} = \left[\overline{OP} \frac{\nabla(t,\phi)}{\nabla_0} - \overline{ON} \right] \tag{4}$$

The dynamic righting arm defined by $\overline{GZ}_{\text{eff}}$ depends on both instantaneous arm lengths as well as instantaneous underwater displacement. For convenience we shall drop the subscript and refer to the dynamic restoring arm as $\overline{GZ}$ instead of $\overline{GZ}_{\text{eff}}$. Substituting (3) into (1) and dividing by $[I_{44} + A_{44}(\infty)]$ leads to the non-dimensional form of the roll equation of motion given by (5).

$$\ddot{\phi} + \int_{-\infty}^{t} k_{44}(t-\tau)\dot{\phi}(\tau)d\tau + b_1\dot{\phi} + b_2\dot{\phi}|\dot{\phi}| + c_{44}\overline{GZ}(t,\phi) = 0 \tag{5}$$

where

$$k_{44}(\tau) = \frac{K_{44}(\tau)}{[I_{44} + A_{44}(\infty)]} \qquad b_1 = \frac{B_1}{[I_{44} + A_{44}(\infty)]}$$
$$b_2 = \frac{B_2}{[I_{44} + A_{44}(\infty)]} \qquad c_{44} = \frac{\rho g \nabla_0}{[I_{44} + A_{44}(\infty)]}$$

In (5), the parameters $k_{44}(\tau)$, b_1, b_2 and c_{44} are known values. For a complete analytical description $\overline{GZ}(t,\phi)$ needs to be analytically expressed in terms of parameters which are dependent on the geometry of the hull form.

The wave elevation in irregular head seas ($\beta = 180^0$) can be expressed as a superposition of linear regular components as shown in (6) where a_m, k_m, ω_m and ϵ_m represent the amplitude, wave number, encounter frequency and phase of the m^{th} regular wave component.

$$\eta(t,x) = \frac{1}{2}\sum_{m=1}^{N} a_m \left[e^{i(k_m x + \omega_m t + \epsilon_m)} + e^{-i(k_m x + \omega_m t + \epsilon_m)}\right] \tag{6}$$

Assuming that $\xi_3(\omega_m)$ and $\xi_5(\omega_m)$ are the heave and pitch response amplitude operators (RAO) at encounter frequency ω_m, the corresponding relative wave elevation incorporating the dynamic heave and pitch of the vessel can be expressed as shown in (7).

$$r(t,x) = \frac{1}{2}\sum_{m=1}^{N} a_m \left[\nu(\omega_m, x)e^{i(\omega_m t + \epsilon_m)} + \bar{\nu}(\omega_m, x)e^{-i(\omega_m t + \epsilon_m)}\right] \tag{7}$$

where $(\bar{\ })$ denotes the complex conjugate and $\nu(x,\omega_m)$ and $\bar{\nu}(x,\omega_m)$ are given by

$$\nu(x,\omega_m) = \left[e^{ik_m x} - \xi_3(\omega_m) + x\xi_5(\omega_m)\right] \tag{8}$$
$$\bar{\nu}(x,\omega_m) = \left[e^{-ik_m x} - \bar{\xi}_3(\omega_m) + x\bar{\xi}_5(\omega_m)\right] \tag{9}$$

4. Volterra Series Method for GZ Variation

This section describes the method of obtaining the $\overline{GZ}$ variation in waves using a Volterra series approach. This is an extension of the Volterra $\overline{GM}$ method proposed by Somayajula and Falzarano (2017) where the linear stiffness term was assumed to change in waves while the cubic and higher order terms were assumed to be invariant with time. Time varying $\overline{GZ}$ can be expressed as shown in (10) where the explicit dependence with respect to time t and instantaneous roll angle ϕ is shown.

$$\overline{GZ} = \overline{GZ}(t,\phi) \tag{10}$$

The ship is divided into a set of transverse sections which allows expressing $\overline{GZ}(t,\phi)$ in terms of $\overline{GZ}^{2D}(t,x,\phi)$ as shown in (11) where ∇ is the displacement under the calm waterline in upright condition and $A(x)$ is the sectional area under the calm waterline at the section located x meters from midship in the upright condition.

$$\overline{GZ}(t,\phi) = \frac{1}{\nabla}\int_L \overline{GZ}^{2D}(t,x,\phi)A(x)dx \tag{11}$$

It can be seen from (11) that $\overline{GZ}^{2D}$ depends explicitly on time t. However, at the local section located at a distance x from midship, the time dependence of $\overline{GZ}^{2D}$ is due the change of instantaneous draft at the section with time. Thus, without loss of generality, the time dependence can be expressed in terms of the local draft as shown below

$$\overline{GZ}(t,\phi) = \frac{1}{\nabla}\int_L \overline{GZ}^{2D}(T+r(t,x),x,\phi)A(x)dx \tag{12}$$

where T is the mean draft in calm water and $r(t,x)$ is the relative waterline as defined in (7). Assuming a separation of variables, $\overline{GZ}^{2D}$ can be expressed as an odd polynomial of roll angle ϕ with coefficients which are functions x and $r(t,x)$ as shown in (13).

$$\begin{aligned}\overline{GZ}^{2D}(T+r(t,x),x,\phi) = k_1(T+r(t,x),x)\phi + k_3(T+r(t,x),x)\phi^3\\ + k_5(T+r(t,x),x)\phi^5 + ...\end{aligned} \tag{13}$$

Substituting (13) into (12) results in polynomial approximation for $\overline{GZ}$

$$\overline{GZ}(t,\phi) = K_1(t)\phi + K_3(t)\phi^3 + K_5(t)\phi^5 + ... \tag{14}$$

where

$$K_1(t) = \left[\frac{1}{\nabla}\int_L k_1(T+r(t,x),x)A(x)dx\right] \tag{15}$$

$$K_3(t) = \left[\frac{1}{\nabla}\int_L k_3(T+r(t,x),x)A(x)dx\right] \tag{16}$$

$$K_5(t) = \left[\frac{1}{\nabla}\int_L k_5(T+r(t,x),x)A(x)dx\right] \tag{17}$$

and so on

Expanding $k_i(T+r(t,x),x)$ for $i=1,3,5,...$ as Taylor series about the mean draft T, $K_i(t)$, the time varying coefficient of ϕ^i in (14), can be expressed as

$$\begin{aligned}K_i(t) =& \frac{1}{\nabla}\int_L k_i(T,x)A(x)dx + \frac{1}{\nabla}\int_L \frac{\partial k_i}{\partial z}(T,x)r(t,x)A(x)dx\\ &+ \frac{1}{\nabla}\int_L \frac{1}{2!}\frac{\partial^2 k_i}{\partial z^2}(T,x)r^2(t,x)A(x)dx + ...\end{aligned} \tag{18}$$

$$K_i(t) = K_i^{(0)} + K_i^{(1)}(t) + K_i^{(2)}(t) + ... \tag{19}$$

where

$$K_i^{(0)} = \frac{1}{\nabla}\int_L k_i(T,x)A(x)dx \tag{20}$$

$$K_i^{(1)}(t) = \frac{1}{\nabla}\int_L \frac{\partial k_i}{\partial z}(T,x)r(t,x)A(x)dx \tag{21}$$

$$K_i^{(2)}(t) = \frac{1}{\nabla}\int_L \frac{1}{2!}\frac{\partial^2 k_i}{\partial z^2}(T,x)r^2(t,x)A(x)dx \tag{22}$$

The functions $k_i(T,x)$, $\frac{\partial k_i}{\partial z}(T,x)$ and $\frac{\partial^2 k_i}{\partial z^2}(T,x)$ depend on the geometry of the ship. $k_i(T,x)$ are obtained by polynomial fitting the $\overline{GZ}^{2D}$ curve at a section located at a longitudinal distance x from the origin with respect to the roll angle ϕ. The derivatives are obtained by applying a 5-point finite differencing scheme applied to $\overline{GZ}^{2D}$ evaluated at different mean drafts.

Substituting the relative wave elevation from (7) in (21) results in an expression for $K_i^{(1)}(t)$ as shown in (23)

$$\begin{aligned}
K_i^{(1)}(t) &= \frac{1}{\nabla}\int_L \frac{\partial k_i}{\partial z}(T,x)r(t,x)A(x)dx \\
&= \frac{1}{2}\sum_{m=1}^{N} a_m \left[\left\{\frac{1}{\nabla}\int_L \frac{\partial k_i}{\partial z}(T,x)\nu(\omega_m,x)dx\right\}e^{i(\omega_m t+\epsilon_m)}\right. \\
&\qquad \left. +\left\{\frac{1}{\nabla}\int_L \frac{\partial k_i}{\partial z}(T,x)\bar{\nu}(\omega_m,x)dx\right\}e^{-i(\omega_m t+\epsilon_m)}\right] \\
&= \frac{1}{2}\sum_{m=1}^{N} a_m \left[f_i(\omega_m)e^{i(\omega_m t+\epsilon_m)} + \bar{f}_i(\omega_m)e^{-i(\omega_m t+\epsilon_m)}\right]
\end{aligned} \tag{23}$$

where $f_i(\omega_e)$ is the first order transfer function given by

$$f_i(\omega_e) = \frac{1}{\nabla}\int_L \frac{\partial k_i}{\partial z}(T,x)\nu(\omega_e,x)dx \tag{24}$$

The above approach is applied to a C11 container ship assumed to have a forward speed of 8 knots. The body plan and particulars of the vessel are shown in Figure 2 and Table 1 respectively. The first order transfer functions $f_i(\omega_i)\quad i = 1,3$ and 5 for the C11 hull form are shown in Figure 3. Substituting the relative wave elevation from (7) in (22) results in an expression for $K_i^{(2)}(t)$

$$K_i^{(2)}(t) = \frac{1}{\nabla}\int_L \frac{1}{2!}\frac{\partial^2 k_i}{\partial z^2}(T,x)r^2(t,x)A(x)dx \tag{25}$$

Table 1.: Details of the C11 hull form

Particulars	**Value**
Length between perpendiculars L_{pp} (m)	262.00
Breadth B (m)	40.00
Depth D (m)	24.45
Mean Draft T (m)	12.32
Displacement $\Delta = \rho\nabla$ (tonnes)	76056.00
Vertical Center of Gravity $\overline{KG}$ (m)	18.32
Metacentric Height $\overline{GM}$ (m)	1.973
Roll Natural Period T_n (sec)	22.78

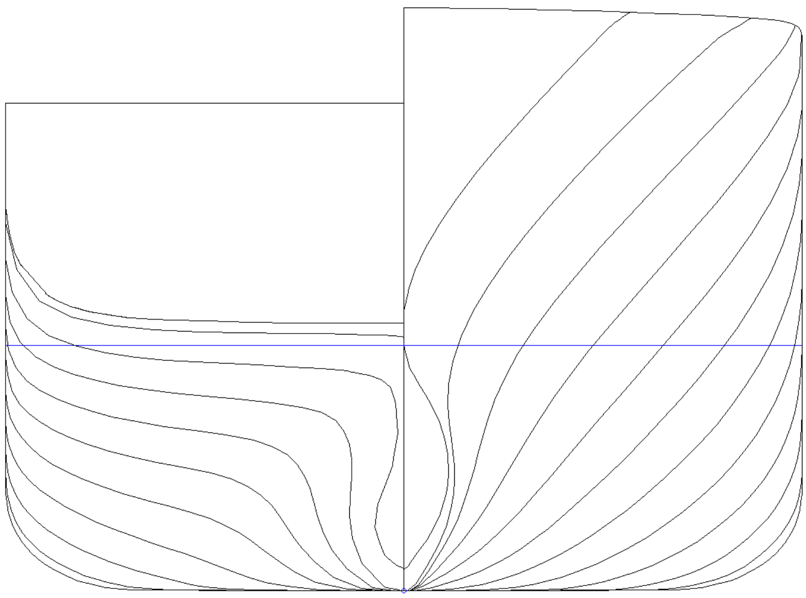
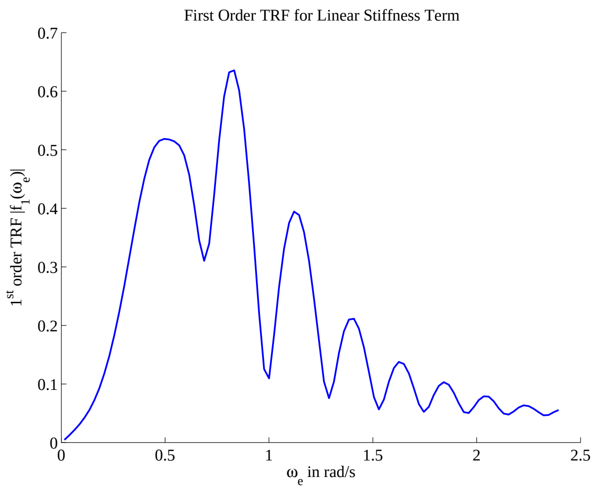
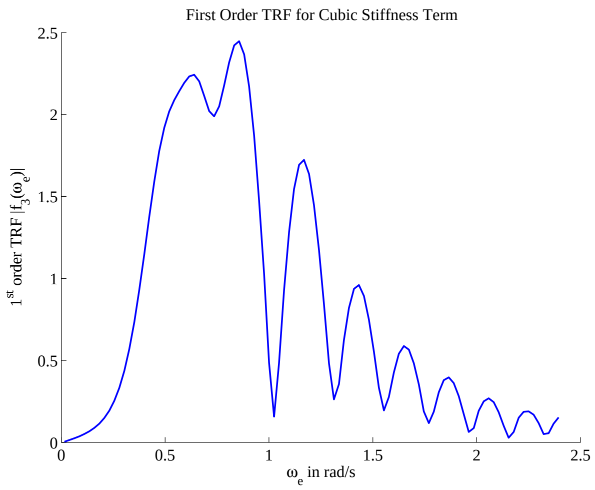


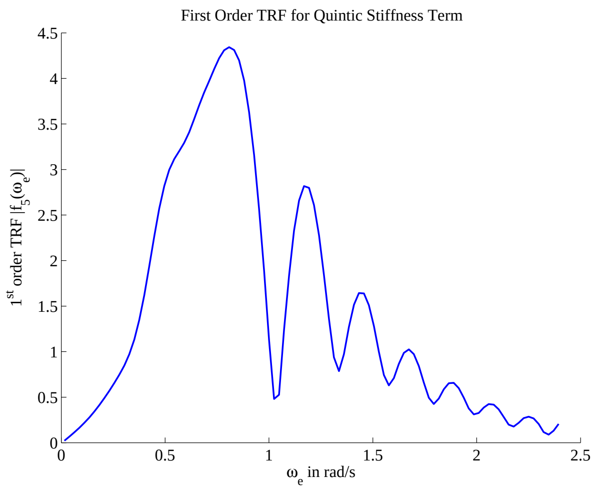
Figure 2.: Body plan of the C11 hull form



(a) Linear Stiffness



(b) Cubic Stiffness



(c) Quintic Stiffness

Figure 3.: First Order Transfer Functions $f_i(\omega_e)$

$$
\begin{aligned}
K_i^{(2)}(t) = \frac{1}{4}\sum_{m=1}^{N}\sum_{n=1}^{N} a_m a_n \Bigg[& \left\{ \int_L \frac{1}{2!}\frac{\partial^2 k_i}{\partial z^2}(T,x)\nu(\omega_m,x)\nu(\omega_n,x)dx \right\} e^{i\{(\omega_m+\omega_n)t+\epsilon_m+\epsilon_n\}} \\
& + \left\{ \int_L \frac{1}{2!}\frac{\partial^2 k_i}{\partial z^2}(T,x)\nu(\omega_m,x)\bar{\nu}(\omega_n,x)dx \right\} e^{i\{(\omega_m-\omega_n)t+\epsilon_m-\epsilon_n\}} \\
& + \left\{ \int_L \frac{1}{2!}\frac{\partial^2 k_i}{\partial z^2}(T,x)\bar{\nu}(\omega_m,x)\nu(\omega_n,x)dx \right\} e^{i\{(-\omega_m+\omega_n)t-\epsilon_m+\epsilon_n\}} \\
& + \left\{ \int_L \frac{1}{2!}\frac{\partial^2 k_i}{\partial z^2}(T,x)\bar{\nu}(\omega_m,x)\bar{\nu}(\omega_n,x)dx \right\} e^{i\{(-\omega_m-\omega_n)t-\epsilon_m-\epsilon_n\}} \Bigg]
\end{aligned} \tag{26}
$$

$$
\begin{aligned}
K_i^{(2)}(t) = \frac{1}{4}\sum_{m=1}^{N}\sum_{n=1}^{N} a_m a_n \Big[& u_i(\omega_m,\omega_n)e^{i\{(\omega_m+\omega_n)t+\epsilon_m+\epsilon_n\}} + \bar{v}_i(\omega_m,\omega_n)e^{i\{(\omega_m-\omega_n)t+\epsilon_m-\epsilon_n\}} \\
& + v_i(\omega_m,\omega_n)e^{i\{(-\omega_m+\omega_n)t-\epsilon_m+\epsilon_n\}} + \bar{u}_i(\omega_m,\omega_n)e^{i\{(-\omega_m-\omega_n)t-\epsilon_m-\epsilon_n\}} \Big]
\end{aligned} \tag{27}
$$

where $u_i(\omega_e,\omega_e)$ and $v_i(\omega_e,\omega_e)$ are second order transfer functions given by

$$
u_i(\omega_m,\omega_n) = \int_L \frac{1}{2!}\frac{\partial^2 k_i}{\partial z^2}(T,x)\nu(\omega_m,x)\nu(\omega_n,x)dx \tag{28}
$$

$$
\bar{v}_i(\omega_m,\omega_n) = \int_L \frac{1}{2!}\frac{\partial^2 k_i}{\partial z^2}(T,x)\nu(\omega_m,x)\bar{\nu}(\omega_n,x)dx \tag{29}
$$

$$
v_i(\omega_m,\omega_n) = \int_L \frac{1}{2!}\frac{\partial^2 k_i}{\partial z^2}(T,x)\bar{\nu}(\omega_m,x)\nu(\omega_n,x)dx \tag{30}
$$

$$
\bar{u}_i(\omega_m,\omega_n) = \int_L \frac{1}{2!}\frac{\partial^2 k_i}{\partial z^2}(T,x)\bar{\nu}(\omega_m,x)\bar{\nu}(\omega_n,x)dx \tag{31}
$$

The second order transfer functions $u_i(\omega_e,\omega_e)$ and $v_i(\omega_e,\omega_e)$ for $i = 1,3$ and 5 are shown in Figure 4. Thus the effective restoring arm in waves $\overline{GZ}(t,\phi)$ is given by (32)

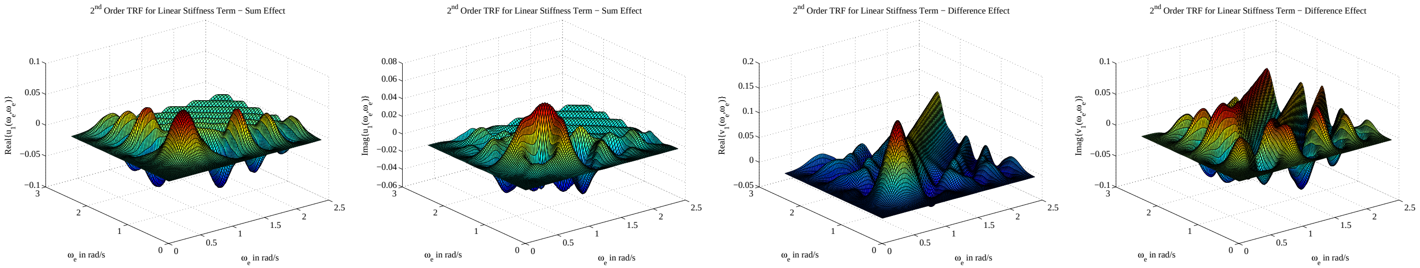
$$
\overline{GZ}(t,\phi) = K_1(t)\phi + K_3(t)\phi^3 + K_5(t)\phi^5 + ... \tag{32}
$$

$$
K_i(t) = K_i^{(0)} + K_i^{(1)}(t) + K_i^{(2)}(t) \tag{33}
$$

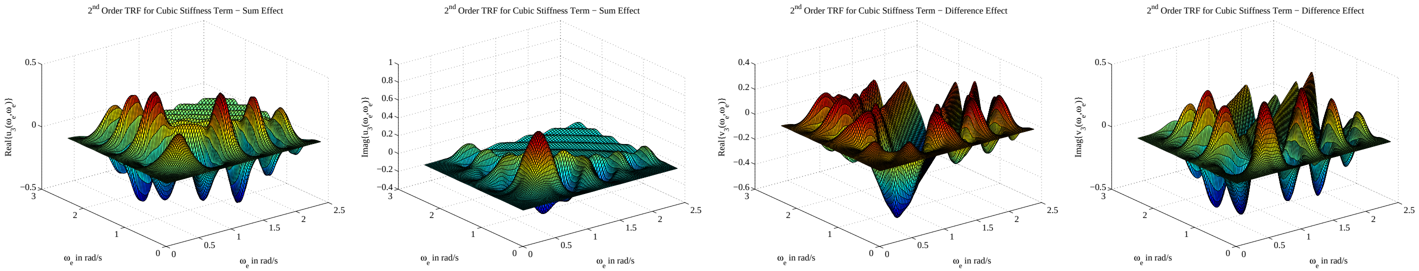
where $K_i^{(0)}$ represent the time invariant coefficients of i^{th} power of ϕ for the calm water $\overline{GZ}$ curve, $K_i^{(1)}(t)$ and $K_i^{(2)}(t)$ represent the time varying change of the coefficients of i^{th} power of ϕ for the effective restoring arm $\overline{GZ}$ in waves and are given by (34) and (35) respectively.

$$
K_i^{(1)}(t) = \frac{1}{2}\sum_{m=1}^{N} a_m \left[f_i(\omega_m)e^{i(\omega_m t+\epsilon_m)} + \bar{f}_i(\omega_m)e^{-i(\omega_m t+\epsilon_m)} \right] \tag{34}
$$

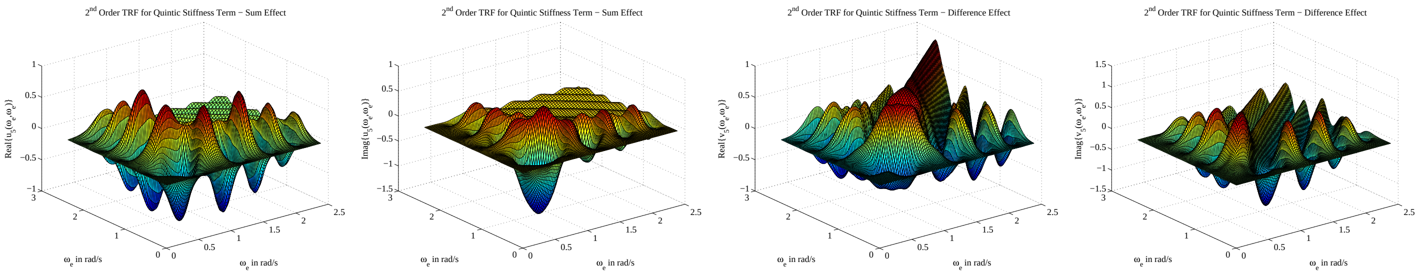
$$
\begin{aligned}
K_i^{(2)}(t) = \frac{1}{4}\sum_{m=1}^{N}\sum_{n=1}^{N} a_m a_n \Big[& u_i(\omega_m,\omega_n)e^{i\{(\omega_m+\omega_n)t+\epsilon_m+\epsilon_n\}} + \bar{v}_i(\omega_m,\omega_n)e^{i\{(\omega_m-\omega_n)t+\epsilon_m-\epsilon_n\}} \\
& + v_i(\omega_m,\omega_n)e^{i\{(-\omega_m+\omega_n)t-\epsilon_m+\epsilon_n\}} + \bar{u}_i(\omega_m,\omega_n)e^{i\{(-\omega_m-\omega_n)t-\epsilon_m-\epsilon_n\}} \Big]
\end{aligned} \tag{35}
$$



(a) $Re\{u_1(\omega_e, \omega_e)\}$ (b) $Im\{u_1(\omega_e, \omega_e)\}$ (c) $Re\{v_1(\omega_e, \omega_e)\}$ (d) $Im\{v_1(\omega_e, \omega_e)\}$



(e) $Re\{u_3(\omega_e, \omega_e)\}$ (f) $Im\{u_3(\omega_e, \omega_e)\}$ (g) $Re\{v_3(\omega_e, \omega_e)\}$ (h) $Im\{v_3(\omega_e, \omega_e)\}$



(i) $Re\{u_5(\omega_e, \omega_e)\}$ (j) $Im\{u_5(\omega_e, \omega_e)\}$ (k) $Re\{v_5(\omega_e, \omega_e)\}$ (l) $Im\{v_5(\omega_e, \omega_e)\}$

Figure 4.: Second order transfer functions $u_i(\omega_e, \omega_e)$ and $v_i(\omega_e, \omega_e)$ for Linear ($i = 1$), Cubic ($i = 3$) and Quintic ($i = 5$) Stiffness Coefficients

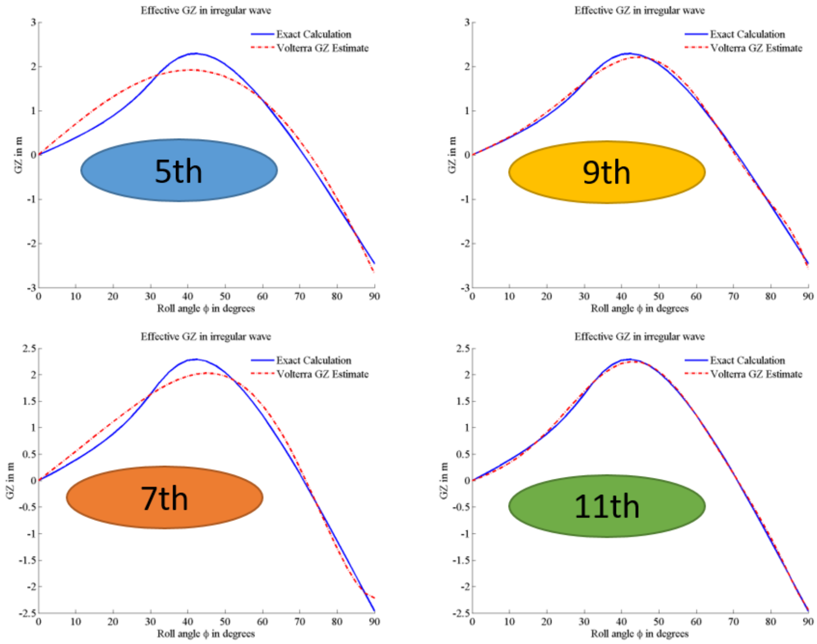


Figure 5.: Capture of instantaneous $\overline{GZ}$ by different orders of Volterra $\overline{GZ}$ model

Generally, it is quite common to approximate the roll restoring arm by a cubic polynomial of roll angle. For wall sided ships, it is well known that the roll restoring arm exhibits an initial hardening for smaller roll angles followed by an eventual softening. Such a characteristic can only be captured by a polynomial of at least 5^{th} order. However, studies by Falzarano (1990) have shown that sometimes even a 7^{th} or even a 9^{th} order polynomial is required to capture the calm water restoring arm. In order to determine the required order to capture the time varying nonlinear stiffness during parametric roll, the $\overline{GZ}$ at an instant in a 3-hour long simulation incident to a Bretschneider spectrum with with $H_s = 5.0$ meters and $T_p = 13.0$ seconds is calculated exactly and compared against various orders of Volterra $\overline{GZ}$ model. It is clearly seen from Figure 5 that both quintic (5^{th} order) and septic (7^{th} order) stiffness models are insufficient in this case to capture the time varying stiffness of C11 hull form in irregular waves. An 11^{th} order polynomial is seen to provide an accurate capture and is used in the further analyses. The effective restoring arm in waves $\overline{GZ}(t, \phi)$ is substituted into (5) and resulting differential equation is solved for the roll motion.

5. Comparison of Single Degree of Freedom Models

The simulated parametric roll motion using the Volterra $\overline{GZ}$ method is compared against a previously validated (Somayajula and Falzarano 2017; Somayajula 2017) nonlinear time domain simulation program - SIMDYN and is shown in Figure 6. Also comparisons are made between the roll motions predicted by Volterra $\overline{GZ}$ method and the following three alternate approaches popular in the literature to model the restoring arm in waves:

1. Volterra $\overline{GM}$ model with time invariant higher order stiffness terms retained from calm water $\overline{GZ}$ curve

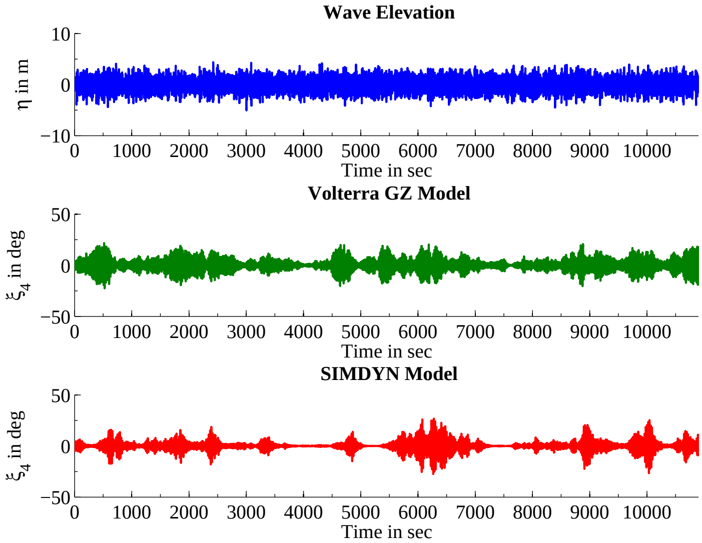


Figure 6.: Comparison of roll motion between "SIMDYN" and "Volterra $\overline{GZ}$ model" of Pram hull form in a 3-hour realization of a Bretschneider spectrum with $H_s = 5.0$ meters and $T_p = 13.0$ seconds

2. Improved Grim effective wave (IGEW) model incorporating dynamic heave and pitch effects
3. Grim effective wave (GEW) model incorporating dynamic heave and pitch effects

The details of the three alternate models have been discussed by Somayajula and Falzarano (2017) and are not repeated here for brevity. The roll response of all the four models to the same three hour long wave elevation record generated from a Bretschneider spectrum with $H_s = 5.0$ meters and $T_p = 13.0$ seconds are simulated and compared against the SIMDYN simulation. A comparison of the roll spectra from each model and SIMDYN are shown in Figure 7. It can be seen that the Volterra $\overline{GM}$ method significantly over-predicts the roll response. Both the IGEW and GEW $\overline{GZ}$ methods (both including dynamic heave and pitch effects) lead to a roll spectrum peak shifted away from the roll spectrum peak obtained from the SIMDYN simulation. Although, Volterra $\overline{GZ}$ method has a peak matching with the SIMDYN simulations but it also shows a larger bandwidth with a significant energy being observed at the peak location of IGEW and GEW models.

Each of the single degree of freedom models differ from SIMDYN in terms of the physics involved in them. SIMDYN, described in Somayajula and Falzarano (2015a); Somayajula (2017), uses Wheeler (1970) stretching to capture the dynamic pressure above the mean water line. However, the single degree of freedom models assume a purely hydrostatic pressure over the instantaneous underwater hull form to calculate the effective restoring arm. As the single degree of freedom models do not include the effect of dynamic pressure in calculation of $\overline{GZ}(t, \phi)$, the resulting roll amplitudes are expected to differ. However, it can also be seen that of the suggested four single degree of freedom models, the Volterra $\overline{GZ}$ model is the closest to SIMDYN simulations in terms of the spectral peak. In order to confirm that this is not a specific example, several random seeds are tested and it is found that in each case the results are similar to as observed in Figure 7.

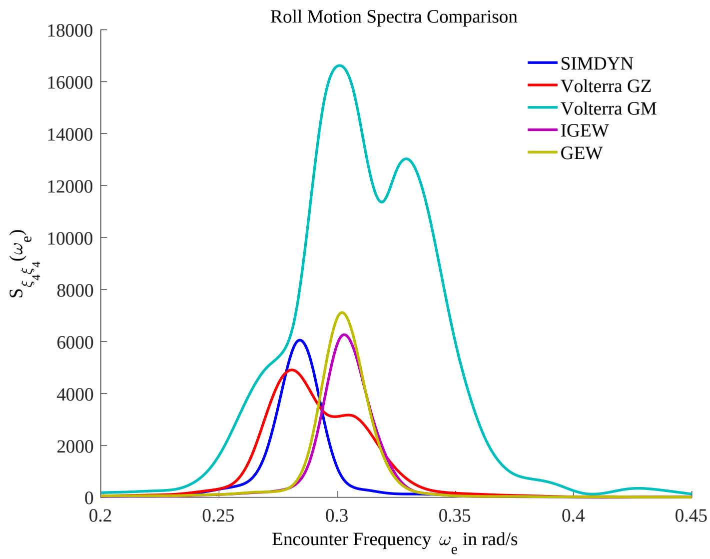


Figure 7.: Comparison of roll spectra using different modeling techniques

6. Conclusion and Future Work

A new analytical model is proposed for parametric roll of ships in irregular head seas which can accurately capture the time variation of both linear ($\overline{GM}$) and nonlinear (cubic, quintic and higher order stiffness components) part of the restoring arm $\overline{GZ}$. The time varying coefficients of various orders of roll stiffness are expressed as a series of Volterra transfer functions which depend on the geometry of the vessel and is hence termed Volterra $\overline{GZ}$ method. A comparison of the proposed model against other existing popular approaches such as the Grim effective wave models is carried out. It is shown that the simulated roll motion spectrum from the proposed model has a much better agreement with multiple degree of freedom coupled nonlinear time domain simulations as compared to the existing models.

The analytical nature of the developed approach allows for a further analysis can be carried out by applying methods from interdisciplinary areas of nonlinear dynamical systems and stochastic dynamics to come up with practical design criteria. These design criteria can potentially be used by designers in the basic design phase to judge the susceptibility of a hull form to the dangerous phenomenon of parametric roll. These design criteria would be in line with the latest developments of the International Maritime Organization's (IMO) second generation level-2 criteria for the assessment of dynamic stability of ships in waves.

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References

Bulian G. 2006. Development of analytical nonlinear models for parametric roll and hydrostatic restoring variations in regular and irregular waves [PhD Dissertation]. Trieste: Università degli studi di Trieste. Available from: https://www.openstarts.units.it/dspace/handle/10077/2518.

Bulian G. 2008. On an improved Grim effective wave. Ocean Engineering. 35(17):1811–1825. Available from: http://www.sciencedirect.com/science/article/pii/S0029801808001923http://dx.doi.org/10.1016/j.oceaneng.2008.08.016.

Bulian G, Francescutto A, Umeda N, Hashimoto H. 2008. Qualitative and quantitative characteristics of parametric ship rolling in random waves in the light of physical model experiments. Ocean Engineering. 35(17-18):1661–1675. Available from: http://linkinghub.elsevier.com/retrieve/pii/S0029801808002011.

Dostal L, Kreuzer E, Sri Namachchivaya N. 2012. Non-standard stochastic averaging of large-amplitude ship rolling in random seas. Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences. 468(April):4146–4173. Available from: http://rspa.royalsocietypublishing.org/cgi/doi/10.1098/rspa.2012.0258.

Falzarano J, Somayajula A, Seah R. 2015. An overview of the prediction methods for roll damping of ships. Ocean Systems Engineering. 5(2):55–76. Available from: http://koreascience.or.kr/journal/view.jsp?kj=TPTPKZ{&}py=2015{&}vnc=v5n2{&}sp=55http://koreascience.or.kr/journal/view.jsp?kj=TPTPKZ{%}7B{&}{%}7Dpy=2015{%}7B{&}{%}7Dvnc=v5n2{%}7B{&}{%}7Dsp=55.

Falzarano JM. 1990. Predicting complicated dynamics leading to vessel capsizing [PhD Dissertation]. Ann Arbor: University of Michigan. Available from: http://deepblue.lib.umich.edu/handle/2027.42/91674.

Grim O. 1961. Beitrag zu dem Problem der Sicherheit des Schiffes im Seegang. Hamburg: Schiffbau-Versuchsanstalt.

Guha A, Falzarano J. 2016. Estimation of hydrodynamic forces and motion of ships with steady forward speed. International Shipbuilding Progress. 62(3-4):113–138. Available from: http://www.medra.org/servlet/aliasResolver?alias=iospress{&}doi=10.3233/ISP-150118.

Hashimoto H, Umeda N, Matsuda A, Nakamura S. 2006. Experimental and numerical studies on parametric roll of a post-panamax container ship in irregular waves. In: Proceedings of the 9th International Conference on Stability of Ships and Ocean Vehicles (STAB'06); vol. 1; Rio de Janerio, Brazil. STAB International Standing Committee. p. 181–190. Available from: http://www.oceanica.ufrj.br/stab2006/files/Abstracts/A090.pdf.

Himeno Y. 1981. Prediction of ship roll damping - A state of the art. Ann Arbor, MI, USA: University of Michigan. Report No:. Available from: http://deepblue.lib.umich.edu/handle/2027.42/91699.

Hua J, Wang W, Chang J. 1999. A representation of GM-variation in waves by the Volterra system. Journal of Marine Science and Technology. 7(2):94–100. Available from: http://jmst.ntou.edu.tw/marine/7-2/94-100.pdf.

Moideen H, Falzarano JM, Sharma S. 2012. Parametric roll of container ships in head waves. Ocean Systems Engineering. 2(4):239–255. Available from: http://koreascience.or.kr/journal/view.jsp?kj=TPTPKZ{%}7B{&}{%}7Dpy=2012{%}7B{&}{%}7Dvnc=v2n4{%}7B{&}{%}7Dsp=239http://koreascience.or.kr/journal/view.jsp?kj=TPTPKZ{&}py=2012{&}vnc=v2n4{&}sp=239.

Moideen H, Somayajula A, Falzarano JM. 2013. Parametric Roll of High Speed Ships in Regular Waves. In: Volume 5: Ocean Engineering; jun; Nantes, France. ASME. p. V005T06A095. Available from: http://proceedings.asmedigitalcollection.asme.org/proceeding.aspx?articleid=1786500http://proceedings.asmedigitalcollection.asme.org/proceeding.aspx?doi=10.1115/OMAE2013-11602.

Moideen H, Somayajula A, Falzarano JM. 2014. Application of Volterra series analysis for parametric rolling in irregular seas. Journal of Ship Research. 58(2):97–105.

Available from: http://www.ingentaconnect.com/content/sname/jsr/pre-prints/content-JOSR130047.

Neves MA, Rodriguez C. 2005. A coupled third order model of roll parametric resonance. In: Soares CG, Garbatov Y, Fonseca N, editors. Maritime transportation and exploitation of ocean and coastal resources - volume 1: Vessels for maritime transportation. London, UK: Taylor and Francis Group Plc; p. 243–253. Available from: http://books.google.com/books?hl=en{&}lr={&}id=bNaIzGwsx18C{&}oi=fnd{&}pg=PA243{&}dq=A+coupled+third+order+model+of+roll+parametric+resonance{&}ots=NUHputyNY8{&}sig=Zkq6{_}yhH2dMVp{_}81ZdEb4RHkMaE.

Neves MA, Rodríguez CA. 2007. Influence of non-linearities on the limits of stability of ships rolling in head seas. Ocean Engineering. 34(11-12):1618–1630. Available from: http://www.sciencedirect.com/science/article/pii/S002980180700039Xhttp://linkinghub.elsevier.com/retrieve/pii/S002980180700039X.

Palmquist M. 1994. On the statistical properties of the metacentric height of ships in following seas. In: 5th Intl Conf on Stability of Ships and Ocean Vehicles; Florida. Available from: http://trid.trb.org/view.aspx?id=455855.

Paulling JR. 1961. The transverse stability of a ship in a longitudinal seaway. Journal of Ship Research. 4(4):37–49.

Somayajula A, Falzarano J. 2015a. Large-amplitude time-domain simulation tool for marine and offshore motion prediction. Marine Systems & Ocean Technology. 10(1):1–17. Available from: http://link.springer.com/10.1007/s40868-015-0002-7.

Somayajula A, Guha A, Falzarano J, Chun HH, Jung KH. 2014. Added resistance and parametric roll prediction as a design criteria for energy efficient ships. Ocean Systems Engineering. 4(2):117–136. Available from: http://koreascience.or.kr/journal/view.jsp?kj=TPTPKZ{&}py=2014{&}vnc=v4n2{&}sp=117http://koreascience.or.kr/journal/view.jsp?kj=TPTPKZ{%}7B{&}{%}7Dpy=2014{%}7B{&}{%}7Dvnc=v4n2{%}7B{&}{%}7Dsp=117.

Somayajula AS. 2017. Reliability Assessment of Hull Forms Susceptible to Parametric Roll in Irregular Seas [PhD Dissertation]. College Station, TX: Texas A&M University.

Somayajula AS, Falzarano J. 2017. A comparative assessment of simplified models for simulating parametric roll. Journal of Offshore Mechanics and Arctic Engineering. 139(2):021103. Available from: http://offshoremechanics.asmedigitalcollection.asme.org/article.aspx?doi=10.1115/1.4034921.

Somayajula AS, Falzarano JM. 2014. Non-linear dynamics of parametric roll of container ship in irregular seas. In: Proceedings of 33rd International Conference on Ocean, Offshore and Arctic Engineering - Volume 7: Ocean Space Utilization; Professor Emeritus J. Randolph Paulling Honoring Symposium on Ocean Technology; jun; San Francisco. ASME. p. V007T12A018. Available from: http://proceedings.asmedigitalcollection.asme.org/proceeding.aspx?doi=10.1115/OMAE2014-24186.

Somayajula AS, Falzarano JM. 2015b. Validation of Volterra series approach for modelling parametric rolling of ships. In: Proceedings of 34th International Conference on Ocean, Offshore and Arctic Engineering - Volume 11: Prof. Robert F. Beck Honoring Symposium on Marine Hydrodynamics; may; St. John's, NL, Canada. ASME. p. V011T12A043. Available from: http://proceedings.asmedigitalcollection.asme.org/proceeding.aspx?doi=10.1115/OMAE2015-41467.

Spyrou K, Tigkas I, Scanferla G, Pallikaropoulos N, Themelis N. 2008. Prediction potential of the parametric rolling behaviour of a post-panamax containership. Ocean Engineering. 35(11-12):1235–1244. Available from: http://linkinghub.elsevier.com/retrieve/pii/S0029801808000747.

Umeda N, Hashimoto H, Vassalos D. 2004. Nonlinear dynamics on parametric roll resonance with realistic numerical modelling. International Shipbuilding Progress. 51(2):205–220. Available from: http://iospress.metapress.com/index/6C37NNM9HJU25AYU.pdf.

Webster W. 1989. Motion in regular waves - Transverse motions. In: Lewis E, editor. Principles of naval architecture vol iii. Jersey City: SNAME.

Wheeler JD. 1970. Method for calculating forces produced by irregular waves. Journal of

Petroleum Technology. 22(3):359–367.